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INTEGRAL EQUATIONS OF THE PROBLEM OF THE TORSION OF AN ELASTIC BODY WITH A THIN DISC-LIKE INCLUSION*

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Non-classical contact conditions are obtained for the axisymmetric torsion problem for an elastic bilayered unbounded medium containing a thin elastic circular inclusion of constant thickness in the connecting plane of the materials. The special singular integral equations of the problem are set up and a method for their approximate solution is described.

The torsion of a bilayered elastic space or half-space containing an elastic inclusion of constant thickness $2H$ in the plane separating the materials is examined in /1, 2/ by reducing the boundary conditions from the inclusion surface to the boundary separating the media (the middle surface of the inclusion) to the accuracy $O(h^2)$ ($h=Ha^{-1}$, where a is the radius of the inclusion) and solving the appropriate system of singular integral equations. By using an operator method and the theory of singular perturbations, this problem is formulated correctly below and its solution is given with an arbitrary number of terms retained in the expansion in powers of the small parameter h.

1. Formulation of the problem. We consider an elastic medium consisting of two half-spaces ($z \lessgtr 0$) with the shear moduli G^1 and G^2, whose plane of separation contains a disc-like elastic inclusion with shear modulus G^0. We assume a concentrated torque M_0 is applied at a distance z_0, $z_0>H$ from the inclusion in the upper half-space ($z\geqslant 0$) on the system axis of symmetry.

The equilibrium equations of a composite layer $0\leqslant \rho<\infty$, $|\zeta|\leqslant h$ under torsion and Hooke's law have the form

$$\partial_\zeta \tau_{\theta z}+\partial_\rho \tau_{r\theta}+2\rho^{-1}\tau_{r\theta}=0 \quad (\rho=r/a, \zeta=z/a) \tag{1.1}$$

$$\tau_{\theta z}=G\partial_\zeta U_\theta, \quad \tau_{r\theta}=G(\partial_\rho-\rho^{-1})U_\theta, \quad U_\theta=a^{-1}u_\theta \tag{1.2}$$

$$\begin{aligned} &G\equiv G(\rho,\zeta)=G_e(\zeta)+[G^0-G_e(\zeta)]U_-(1-\rho)\times \\ &[U_-(\zeta+h)-U_+(\zeta-h)], \quad G_e(\zeta)=G^2+(G^1-G^2)U_-(\zeta) \\ &(G^0, G^1, G^2=\text{const}) \end{aligned} \tag{1.3}$$

Here U_θ, $\tau_{\theta z}$, $\tau_{r\theta}$ are the tangential displacement and the tangential stresses, G is the shear modulus introduced by using asymmetric unit functions $U_\pm(x)$ /3/, and ∂_ρ, ∂_ζ are partial derivatives with respect to ρ, ζ.

We write the relationships (1.1), (1.2) in the matrix-operator form

$$\partial_\zeta f=Af+B \tag{1.4}$$

$$\begin{aligned} &f=\|f_i\|, \ A=\|A_{ij}\|, \ B=\|B_i\| \ (i,j=1,2) \\ &f_1=U_\theta, \ f_2=\tau_{\theta z}, \ A_{11}=A_{22}=B_2=0, \ A_{12}=-G\alpha^2, \ A_{21}= \\ &G^{-1}, \ B_1=-\tau_{r\theta}\partial_\rho \ln G, \ \alpha^2\equiv\partial_\rho^2+\rho^{-1}\partial_\rho-\rho^{-2} \end{aligned} \tag{1.5}$$

Solving (1.4) separately for the inclusion $|\zeta|\leqslant h$, $0\leqslant\rho\leqslant 1$ and the layer outside the inclusion $|\zeta|\leqslant h$, $\rho>1$, we obtain the following functional-operator equation after some reduction

$$\exp(-hA^0)f^0(\rho,h)-\exp(hA^0)f^0(\rho,-h)=\int_{-h}^{h}\exp(-\zeta A^0)B^0 d\zeta, \quad 0\leqslant\rho\leqslant 1 \tag{1.6}$$

$$\begin{aligned} &\exp(-hA^1)f^1(\rho,h)-\exp(hA^2)f^2(\rho,-h)=0, \quad \rho>1 \\ &B_1{}^0=[1-G_e(\zeta)/G^0]\,\tau_{r\theta}(1,\zeta)\,U_-'(1-\rho)\equiv g(\rho,\zeta), \ B_1{}^1=B_1{}^2=0 \end{aligned}$$

where it has been taken into account that $f^1(\rho,0)=f^2(\rho,0)$ for $\rho>1$. Here f^i, $A^i(\rho)$, $B^i(\rho,\zeta)$ ($i=0,1,2$) are functions and differential operators of the form (1.5) defined in the domains of the inclusion and the half-space.

Since /4, 5/

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$$\exp(\pm xA^j) = \cos x\alpha I \pm \alpha^{-1} \sin x\alpha\, A^j$$
$$U_\theta^0(\rho, h) = U_\theta^1(\rho, h), \quad U_\theta^0(\rho, -h) = U_\theta^2(\rho, -h)$$
$$\tau_{\theta z}^0(\rho, h) = \tau_{\theta z}^1(\rho, h), \quad \tau_{\theta z}^0(\rho, -h) = \tau_{\theta z}^2(\rho, -h) \quad (\rho \leqslant 1)$$

where I is the unit matrix, then the following complicated boundary conditions for the problem of the torsion of two elastic media containing a constant-thickness disc-like inclusion in the connecting plane result from (1.6):

$$\cos h\alpha\, [U] + (G_0\alpha)^{-1} \sin h\alpha\, (\tau) = g_1(\rho) \tag{1.7}$$
$$\cos h\alpha\, [\tau] - G_0\alpha \sin h\alpha\, (U) = g_2(\rho) \quad (\rho \leqslant 1)$$
$$\cos h\,\alpha\, [U] + \alpha^{-1} \sin h\alpha\, (\tau_{12}) = 0$$
$$\cos h\alpha\, [\tau] - \alpha \sin h\alpha\, (U_{12}) = 0 \quad (\rho > 1)$$

Here

$$[f] = f^1(\rho, h) - f^2(\rho, -h), \quad (f) = f^1(\rho, h) + f^2(\rho, -h)$$
$$(\tau_{12}) = \frac{1}{G^1}\tau_{\theta z}^1(\rho, h) + \frac{1}{G^2}\tau_{\theta z}^2(\rho, -h)$$
$$(U_{12}) = G^1 U_\theta^1(\rho, h) + G^2 U_\theta^2(\rho, -h)$$
$$g_1(\rho) = -(G^0\alpha)^{-1} \int_{-h}^{h} \sin \zeta\alpha g(\rho, \zeta)\, d\zeta$$
$$g_2(\rho) = \int_{-h}^{h} \cos \zeta\alpha g(\rho, \zeta)\, d\zeta$$

2. Reduction of the problem of integral equations.

Applying the Hankel integral transform /6/, we represent the displacements in the upper and lower half-spaces in the form

$$U_\theta^1 = \int_0^\infty [\varphi_1(\eta) e^{-\eta(\zeta-h)} + \varkappa_0 \eta e^{-\eta|\zeta-\zeta_0|}] J_1(\eta\rho)\, d\eta \quad (\zeta \geqslant 0) \tag{2.1}$$
$$U_\theta^2 = \int_0^\infty \varphi_2(\eta) e^{\eta(\zeta+h)} J_1(\eta\rho)\, d\eta \quad (\zeta \leqslant 0); \quad \varkappa_0 = \frac{M_0}{8\pi G^1 a^3}$$

Taking account of relationships (1.2) and using the spectral property of the Bessel functions $\alpha^2 J_1(\eta\rho) = -\eta^2 J_1(\eta\rho)$ as well as the corollary resulting from this property, we obtain a system of dual integral equations from the boundary conditions (1.7), and when we invert it /6-8/, we arrive at an integral equation in the column-matrix $\Phi(x)$ of the auxiliary functions

$$\int_{-1}^{1} \Phi(t) K(x-t)\, dt = h(x) \quad (|x| \leqslant 1) \tag{2.2}$$
$$K(y) = \| K_{kj}(y) \|, \quad K_{kj}(y) = \int_0^\infty L_{kj}(\eta) \begin{Bmatrix} \sin \eta y \\ \cos \eta y \end{Bmatrix} d\eta \quad \begin{Bmatrix} k = j \\ k \neq j \end{Bmatrix}$$
$$L_{kj}(\eta) = \alpha_{kj} + \beta_{kj} e^{-2\eta\varepsilon}$$
$$h(x) = \| h_j(x) \|, \quad h_1(x) = -2\varkappa_0 x \sum_{n=-1}^{1} \mu_n p(x, \zeta_0 + n\varepsilon)$$
$$h_2(x) = \sum_{n=1}^{2} \Big\{ (-1)^{n+1} 2\varkappa_0 \mu_{-1} q[x, \zeta_0 + (-1)^n \varepsilon] +$$
$$\mu_{n+1} \int_{-1}^{1} \Phi_n(t) Q_n(t, h)\, dt \Big\} + C_1, \quad p(x, \zeta) = \frac{\zeta}{\zeta^2 + x^2}$$
$$q(x, \zeta) = \frac{\zeta^2 - x^2}{(\zeta^2 + x^2)^2}$$
$$Q_1(t, h) + iQ_2(t, h) = l(t) - l(t + i\varepsilon) - \varepsilon \quad (i = \sqrt{-1})$$
$$l(z) = \frac{1 - 2z^2}{\sqrt{1 - z^2}}, \quad \varepsilon = 2h$$
$$\alpha_{11} = \varkappa_3, \; \alpha_{12} = -\beta_{12} = -\varkappa_2, \; \alpha_{21} = -\beta_{21} = \beta, \; \alpha_{22} = -\varkappa_4$$
$$\beta_{11} = \varkappa_1, \; \beta_{22} = \varkappa_5, \; \varkappa_1 = 1 - 2\beta_2\beta_0, \; \varkappa_2 = \beta\beta_0$$
$$\varkappa_3 = 1 + 2\beta_2\beta_0, \; \varkappa_4 = 2\beta_1 + \beta_0, \; \varkappa_5 = 2\beta_1 - \beta_0$$
$$\mu_{-1} = 1 - \beta_0, \; \mu_0 = 2\varkappa_{12}, \; \mu_1 = -1 - (\beta - 2\beta_2)\beta_0$$

$$\mu_2 = -\beta\beta_0 \frac{G^2}{G^0}, \quad \beta_0 = \frac{G^1}{G^0}, \quad \beta = \beta_1 - \beta_2$$
$$\mu_3 = \beta_0 - \beta_2 \left[\left(\frac{G^2}{G^0}\right)^2 + \beta_0^2\right], \quad \beta_j = \frac{G^j}{G^1 + G^2} \quad (k, j = 1, 2)$$

The constant C_1 is determined from the statics conditions of the inclusion. The functions $\varphi_{1,2}$ and $\Phi_{1,2}$ are connected by the relationships

$$\sum_{n=1}^{2} B_{n1}(\eta)\varphi_n(\eta) + \varkappa_0\eta E_1(\eta) = \int_{-1}^{1} \Phi_1(t)\cos\eta t\, dt \tag{2.3}$$
$$\sum_{n=1}^{2} B_{n2}(\eta)\varphi_n(\eta) + \varkappa_0\eta E_2(\eta) = \eta\int_{-1}^{1} \Phi_2(t)\sin\eta t\, dt$$
$$\Phi_1(x) = \Phi_1(-x), \quad \Phi_2(x) = -\Phi_2(-x)$$
$$E_1(\eta) = e^{-\eta\zeta_0}, \quad E_2(\eta) = G^1\eta E_1(\eta)$$
$$B_{n1}(\eta) = (-1)^{n+1}e^{\eta h}, \quad B_{n2}(\eta) = (-1)^n G^n B_{n1}(\eta) \quad (n = 1, 2)$$

Taking account of the shift property of the operator /9/

$$e^{\pm i\varepsilon D_x} e^{i\eta(t-x)} = e^{\pm\eta\varepsilon} e^{i\eta(t-x)} \quad (D_x \equiv d/dx)$$

and also the definition of the Heisenberg function $\delta^+(y)$ /5/

$$\delta^+(y) = \frac{1}{2\pi}\int_0^\infty e^{i\eta y}\, d\eta = \frac{1}{2}\delta(y) + \frac{i}{2\pi}\,\mathrm{V.p.}\frac{1}{y}$$

where $\delta(y)$ is the Dirac function, and V.p. is the symbol of the principal value in the Cauchy sense (we will later omit this symbol), we obtain two identical forms of the very same singular integral equation from (2.2)

$$L_1(\varepsilon D_x)\Phi(x) + \frac{1}{\pi}L_2(\varepsilon D_x)\int_{-1}^{1}\frac{\Phi(t)}{t-x}\, dt = \frac{h(x)}{\pi} \quad (|x| \leqslant 1) \tag{2.4}$$

$$N\Phi(x) + \frac{1}{\pi}M\int_{-1}^{1}\frac{\Phi(t)}{t-x}\, dt + \int_{-1}^{1}\Phi(t)K(t-x, \varepsilon)\, dt = \frac{h(x)}{\pi} \quad (|x| \leqslant 1) \tag{2.5}$$

Here (2.4) is the characteristic singular matrix equation with a given right-hand side (known apart from the constants $h_2(x)$) and with coefficients in the form of the hyperdifferential operators

$$L_j(\varepsilon D_x) = \| L_{nm}^j(\varepsilon D_x)\| \quad (n, m, j = 1, 2) \tag{2.6}$$
$$L_{11}^1 = \varkappa_1 \sin\varepsilon D_x, \quad L_{12}^1 = \varkappa_2(1 - \cos\varepsilon D_x)$$
$$L_{21}^1 = \beta(1 - \cos\varepsilon D_x), \quad L_{22}^1 = -\varkappa_5 \sin\varepsilon D_x$$
$$L_{11}^2 = -\varkappa_2 - \varkappa_1\cos\varepsilon D_x, \quad L_{12}^2 = -\varkappa_2 \sin\varepsilon D_x$$
$$L_{21}^2 = -\beta\sin\varepsilon D_x, \quad L_{22}^2 = -\varkappa_4 + \varkappa_5\cos\varepsilon D_x$$

The second form, Eq.(2.5), is a complete matrix integral Eq.(2.2) with constant coefficients N, M and kernel $K(y, \varepsilon)$

$$N = \begin{Vmatrix} 0 & \varkappa_2 \\ \beta & 0 \end{Vmatrix}, \quad M = \begin{Vmatrix} -\varkappa_3 & 0 \\ 0 & -\varkappa_4 \end{Vmatrix}, \quad K(y, \varepsilon) = \| k_{nm}\| \tag{2.7}$$
$$(n, m = 1, 2)$$
$$k_{11} = -\varkappa_1 k_1, \; k_{12} = -\varkappa_2 k_2, \; k_{21} = -\beta k_2, \; k_{22} = \varkappa_5 k_1$$
$$k_1(y, \varepsilon) = \frac{1}{\pi}\frac{y}{y^2 + \varepsilon^2}, \quad k_2(y, \varepsilon) = \frac{1}{\pi}\frac{\varepsilon}{y^2 + \varepsilon^2}$$

There results from the form of the operator coefficients (2.6) and the special properties of the elements of the matrix-kernel (2.7)

$$k_1(y, \varepsilon) \to \frac{1}{\pi y}, \quad k_2(y, \varepsilon) \to \delta(y) \quad (\varepsilon \to 0) \tag{2.8}$$

that the method of obtaining the complicated contact conditions of two elastic media in terms of an intermediate thin-walled inclusion by retention of a finite number of terms of the expansion of hyperdifferential operators in the small parameter $\varepsilon = 2h$ in relations of the type (1.7) is not totally correct. This is associated with the change in the type of Eq.(2.4) since on truncating the series the coefficients L_1, L_2 become polynomial and the shift property of the operators is lost and, with the fact that the parameter ε in (2.5) can turn out to be large compared with the argument $y = t - x$ in (2.7). Therefore, the assumption on the smallness of the thickness of an elastic coinlike inclusion here has the nature of a singular perturbation and, therefore, appropriate methods /11-13/ must here be applied to solve such kinds of problems.

It should also be noted that on the basis of the Sokhotskii-Plemelj formulas /10/, Eqs. (2.4) and (2.5) are equivalent to a generalized Carleman problem for a rectangle of the diametral section of the inclusion

$$2\pi i \operatorname{Im} [L(\varepsilon D_x)\Phi^+(x)] = h(x) \quad (|x| \leqslant 1) \tag{2.9}$$

that has been studied in only certain special cases* (*Kerekesha, P.V. and Omilio M.A., Investigation of the Carlemann problem for a strip with an analytic shift onward. Dep. No. 2543-79, VINITI, Odessa, 13-07-79, 1979) at this time /14, 15/. Here $\Phi^+(x)$ is the limit value of the function $\Phi(z)$ $(z = x + iy \to x + i0)$ and L is a complex matrix-operator with the elements

$$L_{11} = -\varkappa_3 - i\varkappa_1 e_\varepsilon, \quad L_{12} = \varkappa_2(1 - e_\varepsilon)$$
$$L_{21} = \beta(1 - e_\varepsilon), \quad L_{22} = \varkappa_4 + i\varkappa_5 e_\varepsilon, \quad e_\varepsilon = \exp(i\varepsilon D_x)$$

3. Approximate solution of the integral equations. We consider (2.4) (or (2.5), (2.9)) as /16, 17/

$$P(D_x, \varepsilon)\Phi(x) = 1/\pi\, h(x) \quad (|x| \leqslant 1) \tag{3.1}$$

where $P(D_x, \varepsilon)$ is a pesudodifferential operator with the symbol

$$P(\lambda, \varepsilon) = L_1(i\varepsilon\lambda) + i \operatorname{sgn}\lambda\, L_2(i\varepsilon\lambda) \equiv P_1(\mu) \tag{3.2}$$
$$P_{11} = i \operatorname{sgn}\lambda A_{11}(\lambda, \varepsilon), \quad P_{12} = \varkappa_2 A_{12}(\lambda, \varepsilon)$$
$$P_{21} = \beta A_{21}(\lambda, \varepsilon), \quad P_{22} = i \operatorname{sgn}\lambda A_{22}(\lambda, \varepsilon)$$
$$A_{11}(\lambda, \varepsilon) = -\varkappa_3 - \varkappa_1 e_\varepsilon^0, \quad A_{12}(\lambda, \varepsilon) = 1 - e_\varepsilon^0$$
$$A_{22}(\lambda, \varepsilon) = -\varkappa_4 + \varkappa_5 e_\varepsilon^0, \quad e_\varepsilon^0 = e^{-\varepsilon|\lambda|}, \quad \mu = \lambda/\varepsilon$$

Following /16/, by starting from the properties of the components A_{nm} of the matrix P the solvability of (3.1) can be proved and the uniqueness of the solutions of the singular integral equations presented above can be established in the special class of functions E /16/ in the metric of Sobolev spaces $H_s[-1, 1]$: $\Phi(x)$, $h(x) \in H_s[-1, 1]$ $(s < 0)$.

We will construct the asymptotic form of the solutions as in /16/ according to which

$$\Phi(x) = \sum_{k=0}^{\infty} \varepsilon^{k/2} \sum_{j=0}^{[(k+1)/2]} \ln^j \varepsilon\, [u_{k,j}(x) + \sqrt{\varepsilon}\, v_{k,j}(\xi) + \sqrt{\varepsilon}\, w_{k,j}(\chi)] \tag{3.3}$$

where $\xi = (1 + x)/\varepsilon$, $\chi = (1 - x)/\varepsilon$ are boundary layer variables and $u_{k,j}$, $v_{k,j}$, $w_{k,j}$ are second-order column matrices.

We introduce the operaters H, $\Lambda_{1,2}$

$$Hu(x) \equiv F^{-1}_{\lambda\to x}\{\pi i \operatorname{sgn}\lambda L_2(0)\, u_*(\lambda)\} = L_2(0)\int_{-1}^{1} \frac{u(t)}{t - x}\, dt \tag{3.4}$$

$$\Lambda_1 v(\xi) \equiv F^{-1}_{\mu\to\xi}\{P_1(\mu)\, v_*(\mu)\} = L_1(D_\xi)\, v(\xi) + \frac{1}{\pi} L_2(D_\xi)\int_0^{\infty} \frac{v(s)}{s - \xi}\, ds$$

$$\Lambda_2 w(\chi) \equiv F^{-1}_{\mu\to\chi}\{-P_1(-\mu)\, w_*(\mu)\} = L_1(-D_\chi)\, w(\chi) - \frac{1}{\pi} L_2(-D_\chi)\int_0^{\infty} \frac{w(\tau)}{\tau - \chi}\, d\tau$$

where $F^{-1}\{\ \}$ is the inverse Fourier transform, f_* is the Fourier transform and $i \operatorname{sgn}\lambda L_2(0) = \lim_{\varepsilon\to 0} P(\lambda, \varepsilon)$. Then for the zeroth approximation of the external asymptotic expansion we obtain the singular integral equation

$$L_2(0)\int_{-1}^{1} \frac{u_{0,0}(t)}{t - x}\, dt = h(x) \quad (|x| \leqslant 1) \tag{3.5}$$

$$u_{0,0}(x) = \sqrt{1 - x^2}\,\varphi_0(x), \quad \varphi_0(x) = \frac{2}{\pi^2} L_2^{-1}(0)\int_{-1}^{1} \frac{h(t)\, dt}{(x - t)\sqrt{1 - t^2}}$$

Constructing the residual in the zeroth approximation of the external expansion

$$Q_{0,1}(x) = \frac{1}{\pi} h(x) - P(D_x, \varepsilon)\, u_{0,0}(x) = \sum_{k=0}^{\infty} \varepsilon^k [U_{k,0}^{(0)}(x) + \sqrt{\varepsilon}\, S_{k,0}^{(0)}(\xi) + \sqrt{\varepsilon}\, T_{k,0}^{(0)}(\chi)] \tag{3.6}$$

we obtain the boundary layer functions

$$S_{0,0}^{(0)}(\xi) = -\sqrt{2}\varphi_0(-1)L_1(D_\xi)\sqrt{\xi},\ T_{0,0}^{(0)}(\chi) = -\sqrt{2}\varphi_0(1)L_1(D_\chi)\sqrt{\chi}$$

that belong to the class $G_0^{1/2}$ /16/

$$\{S_{k,0}^{(0)}; T_{k,0}^{(0)}\} \in G_0^{1/2} = \left\{\varphi(\xi)\,|\,\xi > 0,\ \varphi(\xi) = \sum_{k=0}^{\infty}\xi^{-1/2-k}\sum_{j=0}^{k}\varphi_{kj}\ln^j\xi + \sum_{k=1}^{\infty}p_k\xi^{-k};\ \xi\to\infty\right\}$$

Therefore, we obtain the following Wiener-Hopf matrix integral equations

$$\Lambda_1 v_{0,0}^{(0)}(\xi) = S_{0,0}^{(0)}(\xi)\,(\xi > 0),\quad \Lambda_2 w_{0,0}^{(0)}(\chi) = T_{0,0}^{(0)}(\chi)\quad (\chi > 0) \tag{3.7}$$

for the zeroth boundary layer approximations (internal asymptotic expansions).
We find the solution of the first of them in the form

$$v_{0,0}^{(0)}(\xi) = F_{\mu\to\xi}^{-1}P_{1+}^{-1}(\mu)\,\Pi^+P_{1-}^{-1}(\mu)\,F_{\xi\to\mu}lS_{0,0}^{(0)}(\xi)\quad (\xi > 0)$$

where $lS_{0,0}^{(0)}(\xi)$ is the continuation of $S_{0,0}^{(0)}(\xi)$ in the interval $\xi \leqslant 0$, and Π^+ is a convolution operator of the form /16/

$$\Pi^+ v(\mu) = -\frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{v(t)\,dt}{\mu + i0 - t}$$

$P_{1\pm}^{-1}(\mu)$ are matrices inverse to the matrices obtained by factorization of the operator symbol Λ_1 realized by using the methods developed in /18-22/. The solution of the second equation is found in exactly the same way from (3.7).

Having the solutions $u_{0,0}$, $v_{0,0}^{(0)}$, $w_{0,0}^{(0)}$, we again construct the residual to obtain the free terms of the integral Eqs.(3.5), (3.7) with respect to the subsequent approximations in (3.3), and we continue the iteration process in conformity with the scheme in /16, 23/ until the requisite accuracy is achieved. Consequently, by starting from (2.1), (2.3), (3.3), we obtain approximate expressions for the tangential displacements at any point of the elastic half-spaces, including even the boundary of contact between the inclusion and the host. We find appropriate tangential stresses by means of formulas of the type (1.2). The state of stress and strain of the elastic inclusion is determined by solving the matrix-operator Eq.(1.4) for $0 \leqslant \rho \leqslant 1$, $|\zeta| \leqslant h$ taking the stresses and displacements already known on the inclusion end-faces and side surfaces into account.

The solution of problem (3.1) is simplified substantially if the matrix is homogeneous $(\beta \to 0)$, or under the condition that the medium is bilayered but the inclusion is replaced by a slit or an absolutely stiff disc having a finite opening. In these cases the resolving equations become scalar, where the pseudodifferential (scalar) Eqs.(3.1) fall under the procedure of the solution proposed in /16/.

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THE THREE-DIMENSIONAL PROBLEM OF STEADY OSCILLATIONS OF AN ELASTIC HALF-SPACE WITH A SPHERICAL CAVITY*

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The three-dimensional problem of the dynamic theory of elasticity concerning steady harmonic oscillations of an elastic half-space with a spherical cavity is considered. The problem is reduced, with help of the superposition principle, to that of solving a system of six integral equations describing the stress-strain state of the medium. An algorithm for solving the system is given, which can be used in the case when the cavity has a relatively small radius to obtain an approximate solution with any desired degree of accuracy, in the form of an asymptotic expansion. A numerical analysis of the stress-strain state of the elastic medium is given for a wide range of frequencies.

1. Consider the problem of the forced steady harmonic oscillations of an elastic half-space with a deeply placed spherical cavity, in the three-dimensional formulation. The region occupied by the elastic medium is defined by

$$\bar{z} \geqslant 0,\ \bar{r} \geqslant a\ (\sqrt{(\bar{z}+h)^2+x^2+y^2}=\bar{r})$$

where a is the cavity radius, h is the depth of its centre, $\bar{x}, \bar{y}, \bar{z}$ are rectangular Cartesian coordinates and $\bar{r}, \alpha, \beta$ are spherical coordinates attached to the cavity centre. Let x, y, z, r denote the dimensionless coordinates referred to the cavity radius a.

The following boundary conditions are specified at the boundary of the cavity in the general case:

$$\begin{aligned} z=0,\quad & \tau_{xz}=t_1(x,y)e^{-i\omega t},\quad \tau_{yz}=t_2(x,y)e^{-i\omega t},\quad \sigma_z=t_3(x,y)e^{-i\omega t} \\ r=1,\quad & \sigma_r=\tau_1(\alpha,\beta)e^{-i\omega t},\quad \tau_{r\alpha}=\tau_2(\alpha,\beta)e^{-i\omega t},\quad \tau_{r\beta}=\tau_3(\alpha,\beta)e^{-i\omega t} \end{aligned} \tag{1.1}$$

The motion of the medium is described by the dynamic equations of the theory of elasticity in terms of the displalements, i.e. by the Lamé equations /1/.

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